Analysis of a Complex Kind

Week 2

Lecture 3: Iteration of Quadratic Polynomials, Julia Sets

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Quadratic Polynomials

We’ll be looking at polynomials of the form $f(z) = z^2 + c$, where $c \in \mathbb{C}$ is a constant. We’ll study how the behavior of the iterates of $f$ depends on $c$.

- What about other quadratic polynomials? Shouldn’t we be looking, more generally, at $p(z) = az^2 + bz + d$, for constants $a, b, d \in \mathbb{C}$?

- It turns out, for each triple of constants $(a, b, d)$ there is exactly one constant $c$ such that $p(z) = az^2 + bz + d$ and $f(z) = z^2 + c$ “behave the same under iteration”.

- Why? Given $a, b$ and $d$, we define $c = ad + \frac{b}{2} - \left(\frac{b}{2}\right)^2$. Then letting $\varphi(z) = az + \frac{b}{2}$ one can check that $p(z) = \varphi^{-1}(f(\varphi(z)))$ for all $z$. 
\[ p(z) = \varphi^{-1}(f(\varphi(z))) \] for all \( z \).

We write this as \( p = \varphi^{-1} \circ f \circ \varphi \) (read: “phi inverse composed with f composed with phi”). Here is the miracle that happens under iteration:

\[
\begin{align*}
p \circ p & = (\varphi^{-1} \circ f \circ \varphi) \circ (\varphi^{-1} \circ f \circ \varphi) = \varphi^{-1} \circ f \circ f \circ \varphi, \\
p^2 & = \varphi^{-1} \circ f^2 \circ \varphi \\
p^3 & = \varphi^{-1} \circ f^3 \circ \varphi \\
\vdots \\
p^n & = \varphi^{-1} \circ f^n \circ \varphi
\end{align*}
\]

It thus suffices to study the iteration of quadratic polynomials of the form \( f(z) = z^2 + c \).
The Julia Set

- The Julia set (named after the French mathematician Gaston Julia, 1893-1978) of $f(z) = z^2 + c$ is the set of all $z \in \mathbb{C}$ for which the behavior of the iterates is “chaotic” in a neighborhood.

- The Fatou set (named after the French mathematician Pierre Fatou, 1878-1929) is the set of all $z \in \mathbb{C}$ for which the iterates behave “normally” in a neighborhood.

What does this mean??

- The iterates of $f$ behave normally near $z$ if nearby points remain nearby under iteration.

- The iterates of $f$ behave chaotically at $z$ if in any small neighborhood of $z$ the behavior of the iterates depends sensitively on the initial point. We’ll clarify this in examples!
First Example

Let’s look at \( c = 0 \), that is \( f(z) = z^2 \). Then \( f^n(z) = z^{(2^n)} \).

Writing \( z = re^{i\theta} \), we see that \( f^n(z) = r^{(2^n)} \cdot e^{i \cdot 2^n \theta} \). Thus:

- If \( |z| < 1 \), then \( |f^n(z)| = |z|^{(2^n)} \to 0 \) as \( n \to \infty \), so \( f^n(z) \to 0 \) as \( n \to \infty \).
- If \( |z| > 1 \), then \( |f^n(z)| \to \infty \) as \( n \to \infty \), so we say that \( f^n(z) \to \infty \) as \( n \to \infty \).
- If \( |z| = 1 \) then \( z = e^{i\theta} \), so \( f^n(z) = e^{i2^n\theta} \), thus \( |f^n(z)| = 1 \) for all \( n \).
The Julia set of \( f(z) = z^2 \)

We notice: In any little disk around a point \( z \) with \( |z| = 1 \), there are points \( w \) with \( |w| > 1 \) (and for which thus \( f^n(w) \to \infty \)), and other points \( w \) with \( |w| < 1 \) (and for which thus \( f^n(w) \to 0 \)).

The unit circle \( \{ z : |z| = 1 \} \) is thus the locus of chaotic behavior, whereas \( \{ z : |z| > 1 \} \) and \( \{ z : |z| < 1 \} \) form the locus of normal behavior.

We write \( J(f) = \{ z : |z| = 1 \} \) (Julia set) and \( F(f) = \{ z : |z| > 1 \} \cup \{ z : |z| < 1 \} \) (Fatou set).
The Basin of Attraction to $\infty$

More generally, let's look at $f(z) = z^2 + c$. Let

$$A(\infty) = \{z : f^n(z) \to \infty\} \quad \text{“basin of attraction to } \infty \text{”}.$$

**Theorem**

The set $A(\infty)$ is open, connected and unbounded. It is contained in the Fatou set of $f$. The Julia set of $f$ coincides with the boundary of $A(\infty)$, which is a closed and bounded subset of $\mathbb{C}$.

**Recap:**

- The Julia set is a closed and bounded set.
- The Fatou set is open and unbounded and contains $A(\infty)$.
- Also: $J(f) \cap \mathcal{F}(f) = \emptyset$ and both sets are “completely invariant” under $f$, meaning that $f(J) = J$ and $f(\mathcal{F}) = \mathcal{F}$.
Let’s look at another example: $f(z) = z^2 - 2$.
It is hard to calculate and understand the iterates $f^n(z)$!
There is a trick! Conjugate $f$ with

$$\varphi(w) = w + \frac{1}{w}, \varphi: \{w : |w| > 1\} \rightarrow \mathbb{C} \setminus [-2, 2].$$

$f$ maps $[-2, 2]$ to $[-2, 2]$ and $\mathbb{C} \setminus [-2, 2]$ to $\mathbb{C} \setminus [-2, 2]$. We can thus look at

$$\varphi^{-1} \circ f \circ \varphi.$$
Recall: $f(z) = z^2 - 2$, $\varphi(w) = w + \frac{1}{w}$. What is $\varphi^{-1}(f(\varphi(w)))$?

\[ f(\varphi(w)) = (\varphi(w))^2 - 2 \]
\[ = (w + \frac{1}{w})^2 - 2 \]
\[ = w^2 + 2w\frac{1}{w} + \frac{1}{w^2} - 2 \]
\[ = w^2 + \frac{1}{w^2} \]
\[ = \varphi(w^2), \quad \text{so} \]
\[ \varphi^{-1}(f(\varphi(w))) = w^2. \]
Recall: \( f(z) = z^2 - 2 \), \( \varphi(w) = w + \frac{1}{w} \).

\[
\varphi^{-1}(f(\varphi(w))) = w^2, \text{ or } f(z) = \varphi(g(\varphi^{-1}(z))), \text{ where } g(w) = w^2.
\]

Thus, on \( \mathbb{C} \setminus [-2, 2] \), the function \( f(z) = z^2 - 2 \) behaves like \( g(w) = w^2 \) behaves on the exterior of the closed unit disk.

Since the iterates \( g^n(w) \) tend to \( \infty \) for \( |w| > 1 \) we conclude that \( f^n(z) \to \infty \) as \( n \to \infty \) for all \( z \in \mathbb{C} \setminus [-2, 2] \).

Thus \( A(\infty) = \mathbb{C} \setminus [-2, 2] \), and thus \( J(f) = [-2, 2] \).
We have looked at two examples so far and found their Julia sets:

- \( f(z) = z^2 \). We found that \( J(f) = \{z : |z| = 1\} \), the unit circle.

- \( f(z) = z^2 - 2 \). We found that \( J(f) = [-2, 2] \), the closed interval from \(-2\) to \(2\) on the real axis.

These two examples are exceptional in that their Julia sets are “smooth”. In fact, they are the only examples amongst all \( f(z) = z^2 + c \) with smooth Julia sets!

Here are some pictures of other Julia sets. We’ll learn how to create these during the next lecture.